On an arithmetical approach to the Riemann hypothesis

Hisanobu Shinya

July 27, 2009

Abstract

In the paper, we first prove a sufficient condition for the Riemann hypothesis which involves the order of magnitude of the partial sum of the Liouville function. Then we show a formula which is curiously related to the proved sufficient condition.

1 Introduction

In the paper, we are interested in an arithmetical approach to the Riemann hypothesis.

Throughout the paper, the following notations are used.

$\lambda(n) :=$ the Liouville function;

$s(n) := 1$ if n is a square positive integer and $s(n) = 0$ otherwise;

$q(n) := (-1)^{n-1}$;

$Q(x) := \sum_{n \le x} q(n)$;

$L(x) := \sum_{n \le x} \lambda(n)$;

$a * b := \sum_{d|n} a(d) b\left(\frac{n}{d}\right)$;

$\lfloor x \rfloor :=$ the floor function of x;

$\zeta(s) :=$ the classical Riemann zeta function.

Firstly, we will prove the following theorem, which gives a sufficient condition for the Riemann hypothesis.

Theorem 1.1. *The formula*

$$\sum_{n \le x^{1/2}} L\left(\frac{x}{n}\right) = O(x^{3/4}) \tag{1}$$

implies the Riemann hypothesis.

Secondly, as will be seen in the proof of Theorem 1.1, the following formula implies the Riemann hypothesis:

$$\int_1^{x^{1/2}} \frac{L(t)dt}{t^2} = O(x^{-1/4}). \tag{2}$$

Motivated by the relationship between (2) and the Riemann hypothesis, we will prove the following theorem.

Theorem 1.2. *Let a be an arithmetical function. Suppose that*

$$|a(n)| \leq 1, \quad A(x) := \sum_{n \leq x} |a(n)| = O(x^{1/2}).$$

Let $h := q * a$. Then we have

$$\int_1^{x^{1/2}} \frac{h\left(\left\lfloor \frac{x}{t} \right\rfloor\right) L(t)dt}{t^2} = O(x^{-1/4}).$$

Among preliminaries, we cite the following lemma.

Lemma 1.1. *[1, Theorem 3.17] Let f and g be arithmetical functions, and*

$$F(x) := \sum_{n \leq x} f(n), \quad G(x) := \sum_{n \leq x} g(n).$$

If a and b are positive real numbers such that $ab = x$, then

$$\sum_{n \leq x} (f * g)(n) = \sum_{n \leq a} f(n) G\left(\frac{x}{n}\right) + \sum_{n \leq b} g(n) F\left(\frac{x}{n}\right) - F(a)G(b).$$

2 The plan of the proof of Theorem 1.1

In this section, we give an outline of the proof of Theorem 1.1.

Throughout this and the following two sections, we assume (1).

The Riemann zeta function is related to the partial sum of the Liouville function by

$$\frac{\zeta(2s)}{\zeta(s)} = s \int_1^\infty \frac{L(t)dt}{t^{s+1}}. \tag{3}$$

With this formula and a variation of Landau's theorem [1, Chapter 11] on abscissa of convergence of the integral of the form

$$\int_1^\infty \frac{Y(t)dt}{t^s},$$

it is easy to see that the following theorem is sufficient for the Riemann hypothesis.

Theorem 2.1. *The integral*

$$\int_1^x \frac{L(t)dt}{t^{s+1}}$$

converges for $s = 1/2 + \delta$ with each $\delta > 0$.

Important steps of a proof of Theorem 2.1 are described in the following lemmas; Lemmas 2.1 and 2.2 will be used for proving Lemma 2.3, and Lemma 2.3 for proving Lemma 2.4.

Lemma 2.1.

$$\sum_{x^{1/2}<n\leq x} L\left(\frac{x}{n}\right) = O(x^{3/4}).$$

Lemma 2.2.

$$\sum_{x^{1/2}<n\leq x} L\left(\frac{x}{n}\right) = \int_{x^{1/2}}^{x} L\left(\frac{x}{t}\right) dt + O(x^{1/2}).$$

Lemma 2.3.

$$\sum_{n>x} \frac{L(n)}{n^2} = O(x^{-1/2}).$$

Lemma 2.4.

$$L_{3/2}(x) := \sum_{n\leq x} \frac{L(n)}{n^{3/2}} = O(\log x).$$

Theorem 2.1 can be deduced easily from Lemma 2.4 as follows. Fix an arbitrary $\delta > 0$. By [1, Theorem 4.2], we write

$$\sum_{n\leq x} \frac{L(n)}{n^{3/2}} \frac{1}{n^{\delta}} = L_{3/2}(x)x^{-\delta} + \delta \int_1^x \frac{L_{3/2}(t)dt}{t^{\delta+1}}.$$

By Lemma 2.4, it is plain that as $x \to \infty$, the first term on the right side tends to 0 and the integral converges. This shows that the sum on the left converges as $x \to \infty$. Since

$$\begin{aligned}
\left|\sum_{n\leq x} \frac{L(n)}{n^{3/2+\delta}} - \int_1^x \frac{L(t)dt}{t^{3/2+\delta}}\right| &= \left|\int_1^x \frac{L(t)dt}{\lfloor t\rfloor^{3/2+\delta}} + o(1) - \int_1^x \frac{L(t)dt}{t^{3/2+\delta}}\right| \\
&= \left|\int_1^x L(t)\left(\frac{1}{\lfloor t\rfloor^{3/2+\delta}} - \frac{1}{t^{3/2+\delta}}\right) dt + o(1)\right| \\
&\leq \int_1^x t\left|\frac{1}{\lfloor t\rfloor^{3/2+\delta}} - \frac{1}{t^{3/2+\delta}}\right| dt + o(1) \\
&= O\left(\int_1^x \frac{dt}{t^{3/2+\delta}} dt\right),
\end{aligned} \tag{4}$$

we find out that

$$\sum \frac{L(n)}{n^{3/2+\delta}} \quad \text{converges} \Leftrightarrow \quad \int \frac{L(t)dt}{t^{3/2+\delta}} \quad \text{converges.}$$

Because $\delta > 0$ is arbitrary, hence, the theorem follows.

3 Proofs of Lemmas 2.1 and 2.2

In this section, we prove Lemmas 2.1 and 2.2.

By the formula [1, Chapter 2]

$$\sum_{n\le x} L\left(\frac{x}{n}\right) = \lfloor x^{1/2}\rfloor$$

and (1), Lemma 2.1 follows easily.

To prove Lemma 2.2, note that for each $t \in (n, n+1)$, we have

$$\begin{aligned} \left|L\left(\frac{x}{n}\right) - L\left(\frac{x}{t}\right)\right| &= \left|L\left(\frac{x}{n}\right) - \left(\sum_{k\le x/n}\lambda(k) - \sum_{x/t<k\le x/n}\lambda(k)\right)\right| \\ &= \left|\sum_{x/t<k\le x/n}\lambda(k)\right| \\ &\le \sum_{x/t<k\le x/n} 1 \\ &\le \sum_{k\le x/n} 1 - \sum_{k\le x/(n+1)} 1 = \left\lfloor\frac{x}{n}\right\rfloor - \left\lfloor\frac{x}{n+1}\right\rfloor, \end{aligned} \tag{5}$$

and so integrating the left and right sides of the above inequality over the interval $(n, n+1)$ with respect to the variable t, we have

$$L\left(\frac{x}{n}\right) = \int_n^{n+1} L\left(\frac{x}{t}\right) dt + O\left(\left\lfloor\frac{x}{n}\right\rfloor - \left\lfloor\frac{x}{n+1}\right\rfloor\right). \tag{6}$$

If $M = \lfloor x^{1/2}\rfloor + 1$, then

$$\sum_{M\le n\le x} O\left(\left\lfloor\frac{x}{n}\right\rfloor - \left\lfloor\frac{x}{n+1}\right\rfloor\right) = O\left(\sum_{M\le n\le x}\left(\left\lfloor\frac{x}{n}\right\rfloor - \left\lfloor\frac{x}{n+1}\right\rfloor\right)\right) = O(x^{1/2}).$$

Therefore, summing over $M \le n \le x$ in (6), we have

$$\sum_{x^{1/2}<n\le x} L\left(\frac{x}{n}\right) = \int_M^{\lfloor x\rfloor+1} L\left(\frac{x}{t}\right) dt + O(x^{1/2}). \tag{7}$$

Using the trivial estimate $L(x) = O(x)$, it is easy to see that

$$\int_M^{\lfloor x\rfloor+1} L\left(\frac{x}{t}\right) dt - \int_{x^{1/2}}^{x} L\left(\frac{x}{t}\right) dt = O(x^{1/2}).$$

With this estimate and (7), the proof of Lemma 2.2 completes.

4 Proofs of Lemmas 2.3 and 2.4

In this section, we prove Lemmas 2.3 and 2.4, thereby completing the proof of Theorem 1.1.

To prove Lemma 2.3, we make the change of variable $x/t = u$ in the integral

$$\int_{x^{1/2}}^{x} L\left(\frac{x}{t}\right) dt;$$

we have, with Lemmas 2.1 and 2.2,

$$\int_{x^{1/2}}^{x} L\left(\frac{x}{t}\right) dt = x \int_{1}^{x^{1/2}} \frac{L(u)du}{u^2} = O(x^{3/4}),$$

or

$$\int_{1}^{x^{1/2}} \frac{L(u)du}{u^2} = O(x^{-1/4}). \tag{8}$$

By the Prime Number Theorem, we know that the formula (3) is valid for $s = 1$:

$$0 = \int_{1}^{x^{1/2}} \frac{L(u)du}{u^2} + \int_{x^{1/2}}^{\infty} \frac{L(u)du}{u^2},$$

or by (8) and changing $x^{1/2} = y$,

$$\int_{y}^{\infty} \frac{L(u)du}{u^2} = O(y^{-1/2}).$$

The proof of Lemma 2.3 now completes if we use the same estimation, as used in (4), to the pair

$$\int_{y}^{\infty} \frac{L(u)du}{u^2} \quad \text{and} \quad \sum_{n>y} \frac{L(n)}{n^2}.$$

Finally, to prove Lemma 2.4, using [1, Theorem 4.2], we write

$$\sum_{n\le x} \frac{L(n)}{n^{3/2}} = \sum_{n\le x} \frac{L(n)}{n^2} n^{1/2} = L_2(x)x^{1/2} - \frac{1}{2}\int_{1}^{x} \frac{L_2(t)dt}{t^{1/2}}, \tag{9}$$

where

$$L_2(x) := \sum_{n\le x} \frac{L(n)}{n^2}.$$

Since $\lim_{x\to\infty} L_2(x) = A < \infty$ by Lemma 2.3, we write

$$L_2(x) + \sum_{n>x} \frac{L(n)}{n^2} = A,$$

or by Lemma 2.3,

$$L_2(x) = A + O(x^{-1/2}).$$

Substituting this in (9), we get

$$\sum_{n\le x}\frac{L(n)}{n^{3/2}} = O(1)+O\left(\int_1^x t^{-1}dt\right) = O(\log x). \tag{10}$$

This completes the proof of Lemma 2.4.

5 Proof of Theorem 1.2

Next, we prove Theorem 1.2.

The notations have the same meanings as defined in the hypothesis of the theorem.

We let

$$H(x) := \sum_{n\le x}(q*a)(n).$$

Lemma 5.1.

$$H(x) := \sum_{n\le x}(q*a)(n) = O(x^{1/2}).$$

Proof. In Lemma 1.1, choose $f(n)=q(n)$, $g(n)=a(n)$, and $a=x$, $b=1$. Then

$$H(x) = \sum_{n\le x}(a*q)(n) = \sum_{n\le x} a(n)Q\left(\frac{x}{n}\right). \tag{11}$$

Since

$$Q(k) = \begin{cases} 1 & : k \text{ is odd} \\ 0 & : \text{otherwise,} \end{cases}$$

the lemma follows readily. □

Lemma 5.2.

$$(\lambda * q)(n) = \begin{cases} s(n) & : n \textit{ is odd} \\ -2 & : \ n = 2^k r^2,\ r^2 \textit{ is odd and } k \textit{ is odd} \\ 1 & : \ n = 2^k r^2,\ r^2 \textit{ is odd and } k \textit{ is even} \\ 0 & : \textit{otherwise.} \end{cases}$$

Proof. We recall [1, Theorem 2.19]

$$\sum_{d|n}\lambda(d) = s(n). \tag{12}$$

If n is odd, then $q(d)=1$ for all divisors d of n. Hence, the first case of the lemma follows from (12).

Next, suppose that $n=2^k w$, where w is any odd positive integer. Since $q(n)$ is multiplicative and the Dirichlet product of multiplicative functions is

multiplicative [1, Theorem 2.14], it follows that $(\lambda * q)(n)$ is multiplicative. Thus we have

$$(\lambda * q)(2^k w) = (\lambda * q)(2^k)(\lambda * q)(w).$$

If w is not square, then it follows that

$$(\lambda * q)(2^k w) = 0.$$

If $k = 1$

$$(\lambda * q)(2) = \sum_{i=0}^{1} \lambda(2^i)(-1)^{2^{1-i}-1} = -2,$$

and if $k \geq 2$

$$\begin{aligned}(\lambda * q)(2^k) &= \sum_{i=0}^{k} \lambda(2^i)(-1)^{2^{k-i}-1} \\ &= (-1) + (-1)^k + \sum_{i=1}^{k-1}(-1)^i(-1)^{2^{k-i}-1} \\ &= (-1) + (-1)^k + Q(k-1).\end{aligned}$$

This completes the proof of the lemma. □

With the lemmas above, the proof of Theorem 1.2 is completed as follows.

In Lemma 1.1, choose $f(n) = (\lambda * q)(n)$, $g(n) = a(n)$, $a = x$, and $b = 1$. Then using Lemma 5.2, we have

$$\sum_{n \leq x}((\lambda * q) * a)(n) = \sum_{n \leq x}(\lambda * q)(n)A\left(\frac{x}{n}\right) = O(x^{1/2}). \tag{13}$$

On the other hand, since the Dirichlet product is associative [1, Theorem 2.6], we have

$$\sum_{n \leq x}((\lambda * q) * a)(n) = \sum_{n \leq x}(\lambda * (q * a))(n),$$

and so choosing $f(n) = \lambda(n)$, $g(n) = (q * a)(n)$, and $a = b = x^{1/2}$ in Lemma 1.1 gives

$$\begin{aligned}\sum_{n \leq x}((\lambda * q) * a)(n) &= \sum_{n \leq x}(\lambda * (q * a))(n) \\ &= \sum_{n \leq x^{1/2}} \lambda(n)H\left(\frac{x}{n}\right) + \sum_{n \leq x^{1/2}} (q * a)(n)L\left(\frac{x}{n}\right) - L(x^{1/2})H(x^{1/2}).\end{aligned} \tag{14}$$

Using $L(x) = O(x)$ and Lemma 5.1, we get

$$\sum_{n \leq x^{1/2}} \lambda(n)H\left(\frac{x}{n}\right) = O\left(\sum_{n \leq x^{1/2}} \frac{x^{1/2}}{n^{1/2}}\right) = O(x^{3/4}) \tag{15}$$

and

$$L(x^{1/2})H(x^{1/2}) = O(x^{3/4}). \tag{16}$$

Combining (13), (14), (15), and (16), we have

$$\sum_{n\le x^{1/2}} h(n)L\left(\frac{x}{n}\right) = O(x^{3/4}).$$

The rest of the proof is similar to the steps by which we obtained (2) from (1). We only give a sketch.

By (13), we have

$$\sum_{x^{1/2}<n\le x} h(n)L\left(\frac{x}{n}\right) = O(x^{3/4}). \tag{17}$$

Then we use the estimation same as (5) to the pair $h(n)L(x/n)$ and $h(n)L(x/t)$. When we sum over, we use, instead of $h(n)$,

$$h_x := \max_{x^{1/2}<n\le x} |h(n)|$$

so that the absolute error between $\sum_{x^{1/2}<n\le x} h(n)L(x/n)$ and $\int_M^{\lfloor x\rfloor+1} h(\lfloor t\rfloor)L(x/t)dt$ is at most $h_x O(x^{1/2})$. From the hypothesis, it is plain that $h_x = O(x^\epsilon)$ for each $\epsilon > 0$. Thus

$$\sum_{x^{1/2}<n\le x} h(n)L\left(\frac{x}{n}\right) = \int_M^{\lfloor x\rfloor+1} h(\lfloor t\rfloor)L\left(\frac{x}{t}\right)dt + O(x^{1/2+\epsilon}).$$

The equation (17) and the similar analysis gives

$$\int_{x^{1/2}}^{x} h(\lfloor t\rfloor)L\left(\frac{x}{t}\right)dt = O(x^{3/4}),$$

and the change of variable $x/t = u$ gives the desired result.

6 Remarks on Theorem 1.2

We end this discussion with some observations on Theorem 1.2.

1. It is easy to see that the method works pretty well with the Möbius function, instead of $\lambda(n)$, as well. That is, if we let $M(x)$ denote the partial sum of the Möbius function, then

$$\int_1^{x^{1/2}} \frac{h\left(\left\lfloor \frac{x}{t}\right\rfloor\right)M(t)dt}{t^2} = O(x^{-1/4}).$$

Although this type of formulas are curious when compared with (2), it is not sure if the rate of vanishing is necessarily $O(x^{-1/4})$. This rate stems essentially from the partial sum

$$\sum_{n\le x^{1/2}} \lambda(n)H\left(\frac{x}{n}\right) = O(x^{3/4}).$$

While we may infer from the flexibility for the choice of the arithmetical function a that the rate $O(x^{3/4})$ for the partial sum is a necessity, there is one persuasive argument for the negative case. Consider the sum

$$\sum_{n \le x^{1/2}} \lambda(n) \left\lfloor \frac{x^{1/2}}{n^{1/2}} \right\rfloor .$$

The factor $\lfloor x^{1/2}/n^{1/2} \rfloor$ is seen as the partial sum $\sum_{k \le x/n} s(k)$, and we may write

$$\sum_{n \le x^{1/2}} \lambda(n) \left\lfloor \frac{x^{1/2}}{n^{1/2}} \right\rfloor = x^{1/2} \sum_{n \le x^{1/2}} \frac{\lambda(n)}{n^{1/2}} + O(x^{1/2}).$$

On the Riemann hypothesis, it is easy to show that

$$\sum_{n \le x^{1/2}} \frac{\lambda(n)}{n^{1/2}} = O(x^{\epsilon}) \quad \text{for each } \epsilon > 0,$$

and so we have

$$\sum_{n \le x^{1/2}} \lambda(n) \left\lfloor \frac{x^{1/2}}{n^{1/2}} \right\rfloor = O(x^{1/2+\epsilon}).$$

2. The density of the nonzero arguments of $h(n)$ less than a given $Y > 0$ is an important factor when we talk of usefulness of Theorem 1.2. For instance, consider the sum

$$\sum_{n \le x^{1/2}} s(n) F\left(\frac{x}{n}\right),$$

where $F(y) = \lfloor y^{1/2} \rfloor$. It is readily seen that the sum has the order $O(x^{1/2} \log x)$;

$$\sum_{n \le x^{1/2}} s(n) \left\lfloor \frac{x^{1/2}}{n^{1/2}} \right\rfloor = \sum_{n \le x^{1/4}} \left\lfloor \frac{x^{1/2}}{n} \right\rfloor = O\left(\sum_{n \le x^{1/4}} \frac{x^{1/2}}{n} \right).$$

If we apply the technique in the proof of Theorem 1.1 to the sum, we have

$$\int_1^{x^{1/2}} \frac{s\left(\left\lfloor \frac{x}{t} \right\rfloor\right) F(t) dt}{t^2} = O(x^{-1/2} \log x).$$

Note that the integral is vanishing, although the integrand is nonnegative. If we choose, for example, $a(n) = s(n)$ in Theorem 1.2, then the resulting function $(q * s)(n)$ is zero only if n is divisible by 2^2 (this follows easily from the fact that $(q * s)(n)$ is multiplicative).

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1976.